Estimating US Equity and Bond Risk Premiums using a Quadratic Gaussian Joint Pricing Model

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Abstract

Kikuchi[2016] proposes a joint pricing model for stocks and bonds in a no-arbitrage framework. His model provides us with a representation of stock prices that is consistent with the quadratic Gaussian term structure interest rate model, in which the short rate is defined as a quadratic form of the state variables. In this model, the stock price is represented as an exponential-quadratic of the state variables and becomes well-defined under certain sufficient conditions. In this article, we empirically analyze U.S. data using the model proposed by Kikuchi[2016]. We estimate the risk premiums for stocks and bonds and analyze not only the effects of the FED’s quantitative easing policy (QE) but also how QE’s termination affected these premiums.

Key Words: risk premium, quadratic Gaussian term structure model, unscented Kalman filter, algebraic Riccati equation, controllability

JEL Classification: C13, E43, E44, G12

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1. Introduction

In financial business practice, it is important to simultaneously estimate risk premiums of multiple financial assets. For example, investors need to estimate risk premiums to optimize their portfolios. Furthermore, in recent years, central banks in developed countries, such as the Federal Reserve Board (FED), the European Central Bank (ECB), and the Bank of Japan (BOJ), have paid attention to risk premium developments with regard to multiple assets such as longer-term government bonds, equities, and defaultable bonds. These central banks began conducting quantitative easing following the backdrop of the collapse of Lehman Brothers in September 2008. They intended to lower not only the risk premiums of longer-term bonds, called term premiums, but also those of riskier assets through large scale longer-term government bond purchases. Simultaneous estimates of risk premiums of several asset classes are essential for both investors and central banks; therefore, simultaneous estimates of stock and bond risk premiums are appropriate starting points.

Which models work best in simultaneously estimating risk premiums of stocks and bonds? A unified pricing framework that jointly deals with these assets’ prices is desirable. One of the likely candidates for a unified pricing framework is a no-arbitrage framework. With respect to estimates of term premiums, the term structure model based on a no-arbitrage framework has been widely used in previous studies. Many of these studies have achieved a good fit with market bond prices. Conversely, studies on stocks under a no-arbitrage framework are limited.

Until now, Bekaert and Grenadier [2002], Mamaysky [2002], d’Addona and Kind [2006], Lemke and Werner [2008], and Bäuerle and Pfeiffer [2013] have worked on joint pricing models for stocks and bonds under a no-arbitrage framework. Except for the study by Bäuerle and Pfeiffer [2013], the studies have been based on affine Gaussian frameworks, and they obtain the stock price representation in a manner that is consistent with the affine Gaussian term structure model. For example, Mamaysky [2002] provides stock price representations as exponential-affine of the Gaussian state variable \( X_t \), multiplied by an affine function of \( X_t \). In the model, dividend yield is obtained as an affine function of \( X_t \). Here, we touch upon a problem with the affine Gaussian joint pricing model’s fit to the market data. Since the financial crisis of 2008, short-term nominal interest rates in developed countries have stayed near zero. Therefore, we need a model that can capture yield curves when short-term interest rates remain at this low level. However, the affine Gaussian joint pricing model does not ensure yield curves with near-zero short-term interest rates. Furthermore, the model’s future short-term interest rates could have large negative values, which could lead to inaccurate term premium
estimates. Moreover, dividend yields in this model could also be negative.

As indicated above, to address this problem with the affine Gaussian joint pricing model, what is preferred is a term structure interest rate model, where short-term interest rates cannot have large negative values. The quadratic Gaussian term structure model (QGTM), studied by Ahn, Dittmar, and Gallant [2002] and Leippold and Wu [2002], is considered to be one of the most desirable models. In QGTM, interest rates are represented as a quadratic form of the state variables and have a lower bound, for example, as zero. This excludes the possibility of interest rates having large negative values.

In a manner that is consistent with a no-arbitrage condition, Kikuchi [2016] proposed a joint pricing model of stocks and bonds, by incorporating QGTM into the dividend discounted cash flow pricing model of stocks. Under certain conditions, the stock price is represented as an exponential-quadratic form of the state variables $X_t$, multiplied by the quadratic form of $X_t$. Consequently, the dividend yield becomes a quadratic form of $X_t$, thus creating a positive value. In this study, we empirically analyze U.S. data using the model proposed by Kikuchi [2016]. We estimate and analyze risk premiums of stocks and bonds. In particular, we focus on both the FED’s quantitative easing policy (QE) and how QE’s termination affected these premiums.

The rest of the study is organized as follows. In Section 2, we present the theoretical basis of Kikuchi [2016], which includes the set-up, bond pricing, and stock pricing elements of the model. In Section 3, we explain the estimation methodology. Section 4 presents the estimation results. The conclusions are presented in Section 5.

2. Theory

2.1 Setup

In this study, we fix a probability space $(\Omega, F, \mathbb{F}, \mathbb{P})$ that satisfies the usual condition. Here, let $\mathbb{P}$ denote the physical measure. In addition, we assume that the market is complete, so that the risk-neutral measure, $\mathbb{Q}$, uniquely exists. Let’s consider a state variable $X_t$, following the Ornstein-Uhlenbeck process under the physical measure $\mathbb{P}$. That is shown as follows:

$$dX_t = K_t^P (\theta_t^P - X_t)dt + \Sigma_t^X dW_{t,t}^P,$$

where $dW_{t,t}^P$ is an $N$ dimensional Brownian motion. Moreover, we assume that $\Sigma_t^X$ is a diagonal matrix with positive diagonal elements.

In addition to $X_t$, we define another state variable $Y_t$ as follows:
\[
dY_t = \mu^P dt + K^P_X X_t dt + \sum_{y,1} dW^P_{t,1} + \sum_{y,2} dW^P_{t,2},
\]

where \( W^P_{t,2} \) is an \( M \) dimensional Brownian motion with \( \text{cov}(W^P_{t,1}, W^P_{t,2}) = 0_{N \times M} \). As can be seen in equation (2), \( Y_t \) is a non-stationary process.

We assume that there exists a market price for the risk of \( W^P_{t,1} \) and \( W^P_{t,2} \). Let \( \Lambda_{t,1} \) be defined such that \( dW^Q_{t,1} = dW^P_{t,1} + \Lambda_{t,1} dt \) where \( W^Q_{t,1} \) is an \( N \) dimensional Brownian motion under the risk-neutral measure. We also assume that \( dW^Q_{t,2} = dW^P_{t,2} + \Lambda_{t,2} dt \) where \( W^Q_{t,2} \) is an \( M \) dimensional Brownian motion under the risk-neutral measure. Particularly, we model \( \Lambda_{t,1} \) as

\[
\Lambda_{t,1} = \lambda_0 + \lambda_1 X_t \quad \text{and} \quad \Lambda_{t,2} = \lambda_1 + \lambda_2 X_t,
\]

according to Duffee [2002]. This is called the essentially affine setting. Describing \( X_t \) under \( Q \) as

\[
dX_t = K^Q_X (\theta^Q - X_t) dt + \sum_{X} dW^Q_{t,1},
\]

we find the following relationships from equations (1) and (3)

\[
K^P_X \theta^P = K^Q_X \theta^Q + \sum_{X} \lambda_0, \quad K^P_X = K^Q_X - \sum_{X} \lambda_1.
\]

Under \( Q \), the non-stationary state process, \( Y_t \), is given by the following:

\[
dY_t = \mu^Q dt + K^Q_Y X_t dt + \sum_{y,1} dW^Q_{t,1} + \sum_{y,2} dW^Q_{t,2}.
\]

Therefore, the essentially affine setting, equations (2), and (5) lead to the following relationships:

\[
\mu^P = \mu^Q + \sum_{y,1} \lambda_0 + \sum_{y,2} \lambda_1, \quad K^P_Y = K^Q_Y + \sum_{y,1} \lambda_1 + \sum_{y,2} \lambda_2.
\]

Next, in this study, the risk-free short rate is defined as a quadratic form of the state variable \( X_t \):

\[
r_t = X_t \Psi X_t,
\]

where the superscript of \( X \) represents the transposition of \( X \), and \( \Psi \) is assumed to be positive definite. This setting ensures the positivity of the short rate \( r_t \). Note that we sometimes denote \( r_t \) by \( r(t, X_t) \) to emphasize that \( r_t \) depends on \( X_t \).

### 2.2 Bond Pricing

Under the setting indicated above, we derive the zero-coupon bond-pricing formula. First,
the zero-coupon bond price \( P^{T,t}(t, X_t) \) at time \( t \) with maturity \( T \) is described as follows:

\[
P^{T,t}(t, X_t) = E_t^Q \left[ \exp \left( -\int_t^T r(u, X_u) du \right) \right]. \tag{8}
\]

We immediately see that the zero-coupon bond price always becomes less than one because the short rate has a positive value from equation (7). This means that zero-coupon yields also have positive values.

Applying the Feynman-Kac theorem to equation (8), we obtain the following partial differential equation for \( P^r(t, X_t) (\tau = T - t) \).

\[
\frac{\partial P^r(t, X_t)}{\partial t} + \kappa^Q(t, X_t) \frac{\partial P^r(t, X_t)}{\partial X_t} - r(t, X_t)P^r(t, X_t) + \frac{1}{2} \text{Tr}(\Sigma X \Sigma' X) \frac{\partial^2 P^r(t, X_t)}{\partial X_t^2} = 0, \tag{9}
\]

\( P^0(T, X_T) = 1 \)

where \( \kappa^Q(t, X_t) = K_X^Q (\Theta^Q - X_t) \).

A candidate of the solution for equation (9) takes the form given by the following:

\[
P^r(t, X_t) = \exp \left( X_t A_t X_t + b_t X_t + c_t \right). \tag{10}
\]

Substituting equation (10) into equation (9), we obtain the differential equations from the conditions that each coefficient corresponding to a degree of \( X_t \) must become equal to zero:

\[
\begin{aligned}
\frac{dA_t}{dt} &= K_X^Q (A_t + A_t) + \Psi - \frac{1}{2} (A_t + A_t) \Sigma X \Sigma' X (A_t + A_t), A_0 = 0_{N \times N}, \\
\frac{db'_t}{dt} &= -(K_X^Q \Theta^Q)' (A_t + A_t) + b_t K_X^Q - b_t \Sigma X - \Sigma (A_t + A_t), b_0 = 0_{N \times 1}, \\
\frac{dc_t}{dt} &= -(K_X^Q \Theta^Q)' b_{t-} - \frac{1}{2} \text{Tr}(\Sigma X \Sigma' X (A_t + A_t) + b_t b_t), c_0 = 0.
\end{aligned} \tag{11}
\]

2.3 Stock Price Representation

We do not address the details of the stock price modeling that is compatible with the term structure interest rate model indicated above under the no-arbitrage condition in this subsection. For further details of the stock modeling, see Kikuchi [2016].

We denote the dividend paid continuously per time to stockholders by \( D(t, Z_t) \), where \( Z_t = (X_t', Y_t') \). We model the dividend \( D(t, Z_t) \) as follows:

\[
D(t, Z_t) = (\delta_c + \delta_t X_t + X_t \Phi X_t) \exp \left( kt + d'X_t + X_t E X_t + c'Y_t \right), \tag{12}
\]

where \( \Phi \) and \( E \) are assumed to be symmetric.

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The following theorem holds true.

**Theorem 1.** We assume the following:

1. \( \Phi \) and \( \Psi - \Phi \) are positive definite,
2. the matrix pair \((K^Q_X, \Sigma_X)\) is controllable;

\[ \delta_0 > \frac{1}{4} \Phi^{-1} \delta. \]

Then, the non-defaultable stock price becomes well-defined and has the following representation:

\[
S(t, Z_t) = \lim_{T \to \infty} E_t^{\mathbb{Q}} \left[ \int_t^T \exp \left( -\int_u^t r(u, X_u) du \right) D(s, Z_s) ds \right] = \exp(kt + d'X_t + X_t E X_t + c' Y_t),
\]

where the coefficients of \( X_t \) and \( Y_t \) satisfy the stock price matrix equation given by the following:

\[
-2E K^Q_X + 2E \Sigma_X \Sigma_X E + \Phi - \Psi = 0_{N \times N},
\]

\[
(-K^Q_X + 2E \Sigma_X \Sigma_X) d + 2E K^Q_X \theta^Q + (2E \Sigma_X \Sigma_X + K^Q_Y) c + \delta_l = 0_{N \times 1},
\]

\[
k + d'K^Q_X \theta^Q + \frac{1}{2} \text{Tr}(\Sigma_X \Sigma_X (2E + dd')) + \frac{1}{2} \text{Tr}(\Sigma_y \Sigma_y c c') + c' \Sigma_y \Sigma_y + c' \mu^Q + \delta_0 = 0.
\]

**Proof.** See Kikuchi [2016] for the proof.

3. Estimation Methodology

3.1 State Space Representation

Moving towards our empirical study, we approximate our continuous-time model to a discrete-time model. We set the unit of time to be one month. We can write the \( X_t \) and \( Y_t \) processes in discrete time as follows:

\[
X_{t+1} = \exp(-K^P_X) X_t + (I - \exp(-K^P_X)) \theta^P + \sqrt{V} \delta^P_{t+1},
\]

\[
Y_{t+1} = Y_t + \mu^P + K^P_Y X_t + \Sigma_{Y \times 2} \delta^P_{2,t+1},
\]

where \( V = \int_0^T e^{K^P_X} \Sigma_X e^{K^P_X} du \) and \( \sqrt{V} \) represents the Cholesky decomposition of \( V \). In addition, \( \delta^P_{1,t+1} \) and \( \delta^P_{2,t+1} \) are independent random variables, each with a standard normal distribution. For the sake of simplicity, we assume that \( \Sigma_{Y \times 1} \) in equation (2) is the zero matrix.

The transition equations consist of the state variable \( X \) and \( Y \). Putting \( X \) and \( Y \) together and calling them \( Z \), we write the transition equation as \( Z_{t+1} = f(Z_t) + \Sigma \delta_{t+1} \).

In our empirical analysis, we let \( N \), the dimension of \( X_t \) be 3, while we let \( M \) be 1. Here, we note that \( X_t \) with three factors is sufficiently flexible to represent the variety of shapes that
the yield curve generally adopts.

The measurement equation is defined by the following:

\[
T_t = \begin{bmatrix}
  z_{yield1}^{(n)} \\
  \vdots \\
  z_{yieldM}^{(n)} \\
  \log S(t) \\
  dyield_t
\end{bmatrix} = \begin{bmatrix}
  g_1(X_t) \\
  \vdots \\
  g_M(X_t) \\
  g_{l+1}(X_t) \\
  g_{l+2}(X_t)
\end{bmatrix} + \begin{bmatrix}
  \eta_{1,t} \\
  \vdots \\
  \eta_{l,t} \\
  \eta_{l+1,t} \\
  \eta_{l+2,t}
\end{bmatrix},
\]

where \( z_{yield1}^{(n)} \) is the zero-coupon yield with \( n \)-month time to maturity at time \( t \), \( \log S(t) \) is the log stock price, and \( dyield_t \) the dividend yield of the stock. Furthermore, \((\eta_{1,t}, \ldots, \eta_{l,t}, \eta_{l+1,t}, \eta_{l+2,t})\) is the measurement error following a multivariate normal distribution. This variance-covariance matrix is given by the following:

\[
Cov(\eta_{1,t}, \ldots, \eta_{l,t}, \eta_{l+1,t}, \eta_{l+2,t}) = \text{diag}(h_1, \ldots, h_1, h_2, h_3) = H.
\]

3.2 Unscented Kalman Filter and Quasi-Maximum Likelihood Method

As we saw in the previous subsection, because \( g_s(X_t) \) is nonlinear, we cannot rely on the Kalman filter to estimate the latent state variable \( X_t \). Therefore, we instead apply the unscented Kalman filter developed by Julier and Uhlmann [1997] to the actual financial market data. We do not cover the unscented Kalman filter algorithm here. Please refer to Julier and Uhlmann [1997].

Given the model parameters, we can estimate the latent state variables according to the unscented Kalman filter.

We estimate the model parameters using the quasi-maximum likelihood method.

\[
\log L(\Theta) = -\frac{NS}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^{S} \left( \log | P_{\gamma_{k+1},\gamma_t} | + (T_k - T_k^-) P_{\gamma_{k+1},\gamma_t}^{-1} (T_k^- - T_k^-) \right),
\]

where \( \Theta \) is a set of model parameters and \( S \) is the number of observation dates.

The optimal model parameter \( \Theta \) is estimated from the maximization of \( \log L(\Theta) \).

3.3 Data

We used monthly data from January 1991 to January 2016 for U.S. zero-coupon yields, the S&P500, and dividend yield of the S&P500. We obtained zero-coupon yields from the FED’s website. The maturities included are for 6 months and 2, 5, 7, and 10 years. Data for the S&P500, and its monthly dividend yield were downloaded from Prof. R. Shiller’s website.
4. Estimation Result

4.1 Model Fit

In this subsection, we illustrate the fit of our model to the actual financial market data. The Appendix presents the estimates of the model parameters and the state variable $X_t$ and $Y_t$. We can examine the performance of our model on the basis of the estimated parameters and the state variables.

Table 1 displays the cross-sectional fitting of zero coupon yields, the dividend yield and the stock price. The mean absolute errors for zero coupon yields are within 20 bps for nearly all the maturities. For the dividend yield, the error value is 3.3 bps. The error for the S&P 500, 2.42%, is provided as the mean absolute error ratio. These estimation results show a good fit to the market data.

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<th>5-year</th>
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Table 1: Mean absolute errors for zero-coupon yields and the dividend yield and the mean absolute error ratio for S&P500. All values are indicated in percentages.

Figure 1 displays the time-series fitting of the estimated zero-coupon yields, dividend yield, and stock price to the market data. This figure shows that the model’s errors over some observations increase in October 2008. This can be attributed to the large price variations in the financial markets during this period. However, overall, the time-series fitting performs well.
4.2 Risk Premium

Figure 2 illustrates estimates of term premiums for the short- and long-term maturity bonds. Focusing on their variations after the beginning of 2009, we find that both premiums have stayed at a very low level. When FED Chairman Bernanke’s speech in May 2013 heightened the market’s concern regarding the reduction of QE3, the estimate for the 10-year term premium rose; however, the degree was slight. Even though QE3 ended in October 2014 and the FED announced an increase in the policy rate in December 2015, we can see that the variation in term premiums did not become large.

Figure 2: Term Premiums

Figure 3 displays equity risk premiums for 1 month, 2 years, 5 years, and 10 years. The equity risk premium is defined as the difference between the n-year (or n-month) expected holding return under P and the n-year (n-month) zero-coupon yield computed using Monte Carlo simulations. According to Figure 3, shorter-term equity risk premiums have higher value than longer-term premiums in close to the same periods. There is the possibility of equity investors being more concerned regarding the short-term risks of equity prices than the long-term risks. Equity risk premiums for all maturities rose sharply at times corresponding to the burst of the dot-com bubble, the financial crisis in 2008, and the European debt crisis.
Furthermore, while the reduction of QE3, its termination, and the lift-off of the policy rate increased the short-term equity risk premium, longer-term premiums were nearly unchanged. It is possible that investors are concerned regarding the development of the FED’s monetary policy at least in the short term.

5. Conclusion

In this study, we conducted an empirical study based on the model proposed by Kikuchi [2016] using U.S. data. We estimated the latent state variables and the model parameters on the basis of the quasi-maximum likelihood method with an unscented Kalman filter. Consequently, we obtained a good fit to the actual financial market data. Furthermore, we computed term premiums and equity risk premiums from the estimated parameters and the state variables. This allows us to analyze the development of these risk premiums and provide a detailed analysis of how the FED’s QE and its termination influenced them. We will continue this work in the near future.

References

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Appendix

Estimates of State Variables and Model Parameters

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