Agent-Based Modeling in Option Pricing under Unknown Volatility and Liquidity Risk

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Abstract

In this paper, we modeled an artificial European option market under unknown volatility with liquidity costs using an agent-based modeling and simulation approach. The option price in the presence of liquidity costs is given by solving a partial differential equation. We proved that both unknown volatility and the unknown drift have significant effects in the pricing bias. Moreover, pricing bias tends to decrease as the drift increasing in the case of low volatility. Our approach may serve as a first step towards the goal of option pricing in disequilibrium with unknown volatility.

Key words: unknown volatility, option pricing, agent-based simulation, liquidity cost.
JEL Classification: C19, G13, G14
1. Introduction

The Black-Scholes model for calculating the premium of an option was introduced in Black et al. (1973). The formula, developed by Black, Scholes and Merton is perhaps the world’s most well-known options pricing model. The model has been based on the simplified assumption of the competitive and frictionless market. The dynamic replication and no-arbitrage pricing paradigms implicit assumes that the risk-free interest rate and the volatility of the underlying asset remain at predetermined and constant levels over the life of the option. Although these simplified models are useful first steps for analyzing markets, however, in reality, neither the model nor the parameter values are known (see Gikhman (2006)). Moreover, the price process may depend on liquidity risk, which is the additional risk in the market due to the timing and size a trade.

In the recent years, various studies have indicated the importance of “model risk” and have emphasized the consequences of neglecting model uncertainty (see Bunnin (2000)). From these perspectives, we need more realistic models without those assumptions above to fit the market phenomena better.

2. Literature Review

From the 1990s to present, the literature on unknown volatility has been developing rapidly. One famous uncertain volatility approach was proposed by Avellaneda et al. (1995), which was assuming that the parameters are uncertain and that the best we can do is to specify a band for each parameter. However, the specification of the volatility range introduces another two unknown parameters. There exists uncertainty over these new unknown parameters estimations. Another important way to deal with uncertainty over model and parameters in the option pricing is using Bayesian approach as in Bunnin (2000). The third related research is micro simulation in the volatility smile research Guo (1998). Finally, a quantitative framework for measuring model uncertainty in the context of derivative pricing was proposed in Cont (2006).

We work in the spirit of Wilmott and Oztukel (1998) to model an artificial European option market under unknown volatility using an agent-based modeling and simulation. We also use the same framework of Çetin et al. (2004) to study how the prices of derivatives should be changed in a financial market with liquidity costs. We show that agent-based modeling and simulation provides a mechanism for conducting experiments that shed light on fundamental properties of the option pricing.

3. Model Assumption

3.1 Research Hypothesis

We assume that people know and utilize Black-Scholes formula to calculate European option price with liquidity costs for its simplicity and popularity. Even if the underlying asset
price process is known with geometric Brownian motion, if nobody really knows the true volatility, it became impossible to make sure profits by creating dynamic risk free portfolios with options and their underlying stocks. Therefore, the main research hypothesis is the difference in market prices with their theoretical option prices in the presence of the unknown volatility. Moreover, we also consider the role of the unknown drift in the option pricing.

3.2 Unknown Volatility

We assume the underlying stock price process follows geometric Brownian motion

\[ dS_t = S_t \mu dt + S_t \sigma_0 dW_t \]

where \( S_t \) is the stock price at time \( t \), \( dt \) is the time step, \( dW_t \) is a standard Brownian motion with mean zero and variance \( dt \), \( \mu \) is the true drift, \( \sigma_0 \) is the true volatility of the returns, \( \mu_0 \) and \( \sigma_0 \) are constant for the life of the option. However, the true drift and volatility of the process are unknown to all market participants. They may not agree with the drift or volatility of the underlying process. Then the believed underlying stock price process for trader \( i \) is

\[ dS_t = S_t \mu_i dt + S_t \sigma_i dW_t \]

With assumption of unknown volatility, each investor has to use his own estimation to pricing options.

Their expectations are influenced by stock market and option market (implied volatility) over time. If investors are aware of their subjectivity on the volatility, their risk attitudes and heterogeneous expectations could become relevant to option trading. Therefore, the fundamental problem lies in that how to model the trader's behavior reasonably with unknown volatility.

Using the framework in Wilmott et al. (1998), we assume investors follow the uncertain volatility approach, thus their volatility estimations include point estimations \( (\sigma_i) \) accompany with the volatility band \( (\sigma_i^{\text{min}}, \sigma_i^{\text{max}}) \).

\[ \sigma_i^{\text{min}} \leq \sigma_i \leq \sigma_i^{\text{max}} \]

Traders are modeled as heterogeneous autonomous agents who are characterized by their “patience” and “judgments”. Patience is the minimum length of days to calculate historical volatility, which is given by a uniform random integer between 5 and 45 (days). For simplicity, we further assume trader's volatility band is given by the corresponding band of historical volatility multiplied by his judgment. For example, if a trader's patience is 10, then his volatility band is calculated with 10 days length price series. Judgment is the trader's subjective adjustment of his experience in historical volatility band, which is given by a uniform random number between 0.5 and 1.5 at the beginning of simulation and remains constant. In this way traders obtain a range of option valuations with their volatility band.
3.3 Model with Liquidity Costs

A basic idea of liquidity costs is: if we buy a larger number of a security, the average buy price will be higher and if we sell a larger number of a security the the average sell price will be lower.

In this study, we are concerned with discrete time hedging and pricing. Let us consider equally spaced times \[0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = T\]. Set \[\Delta t = t_{i+1} - t_i = \frac{T}{n}\] for \[i = 1, \ldots, n\]. We consider the following discrete time version of

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t
\]

where \(Z\) is a standard normal variable, and assume a multiplicative supply curve

\[S(t, x) = f(x)S(t, 0),\]

where \(f\) is a smooth and increasing function with \(f(0) = 1\). Because we consider discrete time trading, the total liquidity costs up to time \(T\) is

\[L_T = \sum_{0 \leq i < T} \Delta X_{u_i} [S(u_i, \Delta X_{u_i}) - S(u_i, 0)],\]

where \(L_{0+} \equiv 0\), and \(L_0 = X_0 [S(0, X_0) - S(0, 0)]\). For the detailed structure of the liquidity costs, see Çetin et al. (2004). Note that the liquidity cost is always non-negative, since \(S(t, x)\) is an increasing function of \(x\). Then, the total liquidity costs up to time \(T\) could rewrite as

\[L_T = \sum_{i=1}^{n} \Delta X_i [S(t_i, \Delta X_i) - S(t_i, 0)] + X_0 [S(0, X_0) - S(0, 0)],\]

where \(\Delta X_i = X_{t_i} - X_{t_{i-1}}\). \(X_t\) represents the trader’s aggregate stock holding at time \(t\) (unit of money market account). Here, \(X_t\) is predictable and optional process with \(X_{0+} \equiv 0\).

We let \(C_0\) denote the value at time 0 of contingent claim \(C\) so that the hedging error inclusive of liquidity costs is

\[H = X_{t_T} (S_{t_T} - S_T) - C + C_0 - L_T - \Delta B,\]

where \(B\) is money market account with

\[\Delta B = rB \Delta t = r(\mathbb{X}_t - C) \Delta t.\]

We aim to find a discrete time hedging strategy that minimizes the expected hedging error. A perfect hedging strategy will produce a zero hedging error with probability 1. But, considering discrete trading and liquidity costs, it is not possible to produce a strategy whose hedging error equals 0. We now find a strategy which is optimal in discrete time with liquidity risk.

An optimal discrete time hedging strategy is a strategy which has expected hedging error is the infinitesimal of the length of the revision interval with order of 3/2, which means

\[E(\Delta H) = O(\Delta t^{3/2}).\]
The total hedging error over the entire interval $[0, T]$ is the sum of $\Delta H$'s. Thus $\mathbb{E}[\Sigma \Delta H] = O(\Delta t^{1/2})$. Therefore, the expected hedging error over the period $[0, T]$ approaches 0 as $t$ becomes small. Moreover, the higher order of $t$ we use, the smaller expected hedging error we get.

When we try to prove that delta hedging is still an optimal strategy. The option price in the presence of liquidity costs is given by solving a partial differential equation

$$C_t + \frac{1}{2} \sigma^2 S^2 C_{SS} + r(C S - C) + f'(0) S^3 C_t^2 C_{SS}^2 = 0, \quad \text{for all } t \in [0, T], S \geq 0,$$

With the terminal condition $C(S,T) = (S - K)^4$.

For more detail of proof, see Section 3 of Tran et al. (2013).

4. Simulation Algorithm

To simulate geometric Brownian motion under the Black-Scholes model, we generate a series of stock price at time $t + \Delta t$ from the formula:

$$S_{t+\Delta t}^i = S_t^i \exp\left(\left(\mu_0 - \frac{\sigma_0^2}{2}\right)\Delta t + \sigma_0 Z_i \sqrt{\Delta t}\right)$$

where $S_t^i$ is the stock price at time $t$, $\sigma_0$ is the stock’s volatility, and $Z_i$ is a standard normal random variable, $i = 1, 2, \ldots, n$. $n$ is the length of the stock market price series. The simulation is organized as two stages:

- On the first stage, only the stock market is opened.
  - Each agent gets his historical volatility estimations using the given 50 days of underlying stock market price series, then he updates them everyday.

- On the second stage, the stock market and the call option market are both opened.
  - There are 50 days for agents to trade call option before expiration date (assume 250 trading days in a year).
  - On each option trading day, traders updates their volatility estimations after a new stock price presented. Then each trader gets his valuation of the call option according to Black and Scholes formula with his estimations and judgments and places a limit order to buy or to short.
  - If the maximum bid quote is not less than the minimum ask quote, trading occurred. The market price is given by the average of maximum bid and the minimum ask. Implied volatility is calculated according to the market price. Thus, each trader updates his volatility estimations from the market implied volatility. Otherwise, no trading occurred. Each trader updates his volatility estimations from implied volatility corresponding to the maximum bid and the minimum ask.

All the simulation is run with 10000 agents and replicates 3000 runs under each condition. To balance the influence of the moneyness on the option price, strike price is set to

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a uniform random number from 0.5 to 1.5 multiply the stock price on the final trading day at each run.

Figure 1 shows a daily time series of option market prices with the theoretical option prices in a typical simulation of 50 days option trading under the condition $\sigma_0$ is 0.3 and the maturity is 0.25 years.

5. Results and Discussion

The first goal of these simulations is to examine the effect of liquidity costs on option prices.

Table 1: Option prices with liquidity costs.

<table>
<thead>
<tr>
<th>Initial spot (0 (Black-Scholes))</th>
<th>Liquidity cost $f'(0)$</th>
<th>$0.0001$</th>
<th>$0.0005$</th>
<th>$0.001$</th>
<th>$0.002$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>1.5617</td>
<td>1.5711</td>
<td>1.6089</td>
<td>1.6662</td>
<td>1.7507</td>
</tr>
<tr>
<td>85</td>
<td>2.7561</td>
<td>2.7698</td>
<td>2.8246</td>
<td>2.8932</td>
<td>3.0303</td>
</tr>
<tr>
<td>90</td>
<td>4.4479</td>
<td>4.4652</td>
<td>4.5343</td>
<td>4.6207</td>
<td>4.7935</td>
</tr>
<tr>
<td>95</td>
<td>6.6696</td>
<td>6.6887</td>
<td>6.7650</td>
<td>6.8604</td>
<td>7.0513</td>
</tr>
</tbody>
</table>

Table 1 presents the option prices inclusive of liquidity costs with varying liquidity costs and initial stock price $S_0$. The parameter values that we used are strike price $K = 100$, $r = 0.2$ and $T = 1$ year. Consistent with intuition, we observe that the option prices increase slightly when the parameter of liquidity costs increases.
Next we check whether the presence of the unknown volatility leads to a significant difference in market prices with their theoretical prices (which are given by Black-Scholes formula using true volatility). We denote the difference between market trading price with BS model as pricing bias or pricing error, which is defined as market price/theoretical price. Option trading prices are obtained from the final trading day at each run and pricing error is measured as the logarithm of market price/theoretical price because of its high volatility.

First, we examined the overall distribution of the pricing error and report the findings in Table 2, which summarizes the descriptive statistics of pricing bias over 3000 replicated simulations under each maturity.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Mean</th>
<th>Median</th>
<th>Maximum</th>
<th>Minimum</th>
<th>Std. Dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Jarque-Bera</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>1.2553</td>
<td>0.5881</td>
<td>0.3561</td>
<td>0.0324</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.6927</td>
<td>0.2656</td>
<td>0.1165</td>
<td>-0.0817</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>11.7054</td>
<td>9.8290</td>
<td>6.8624</td>
<td>2.2410</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>-1.2360</td>
<td>-1.1830</td>
<td>-0.8220</td>
<td>-0.1438</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.3245</td>
<td>2.8523</td>
<td>2.3241</td>
<td>1.5337</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.1126</td>
<td>1.5608</td>
<td>1.3423</td>
<td>0.8473</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>11.6829</td>
<td>11.8595</td>
<td>11.7275</td>
<td>11.0501</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>7330.2</td>
<td>7267.5</td>
<td>7174.4</td>
<td>6861.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
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</tbody>
</table>

Except for the kurtosis, there has been a decline in these statistics as the maturity increasing. Normality assumptions for pricing bias were rejected. As shown in Table 2, median in long maturity (1.25 year) is $-0.082$, indicating an underestimation, on the other hand, we observe an overestimation in the short and middle maturity (from 0.25 to 0.75 year). Volatility is too high, indicating that pricing bias was sensitive to news and random small shocks. Skewness indicates that the distributions of pricing bias were asymmetric, thus there are system bias in compare with Black-Scholes formula in all maturities. Kurtosis are high, indicating pricing bias have a distinct peak near the median, decline rather rapidly, and have heavy tails. These descriptive statistics of pricing bias in Table 2, showed considerable differences in option pricing in compare with Black-Scholes formula in all maturities.

Furthermore, we explore the potential role of the unknown drift in the pricing bias. In Table 3, we report the average pricing error under a wide range of volatilities (from 0.5 to 0.1) and drifts (from $-0.5$ to 0.5) combinations. Average pricing errors are measured as median over 1000 replicated simulations under the same combination of volatility and drift.
As shown in Table 3, for a middle maturity (1.25 year), more than half of the trading prices are lower than the corresponding BS prices under relatively high volatility ($\sigma_0 \geq 0.2$). For example, under the condition $(\mu_0 = -0.5, \sigma_0 = 0.5)$, \begin{align*} 0.5515 \end{align*} in the Table 3 means half of the trading prices are lower than the corresponding BS model prices, and the underestimations are less than 55.15% of the BS model prices. Moreover, we observe a considerable decrease in the pricing error as the drift increasing under low volatility ($\sigma_0 = 0.1$).

In Table 4, we report the average pricing error in a short maturity (0.25 year). These results indicate an increase in pricing error as the volatility decreasing in low drift ($\mu_0 \leq 0.1$). In addition, there is a decline in pricing error as the drift increasing under middle volatility ($\sigma_0 \leq 0.3$).

Finally, to verify the role of the unknown drift in pricing bias, we performed multivariate regression on these data (dependent variable is logarithm of the pricing bias).
In a middle maturity (1.25 year), as shown in Table 5, statistical tests revealed the trading option prices exhibit a significant overestimation (Intercept = 0.6146, \( P < 0.001 \)) and the drift has a significant nonlinear effect in the pricing error.

In a word, contrary to the standard Black and Scholes model with "known" volatility, these results indicate drift plays an important role in option pricing under uncertain volatility with heterogeneous agents. However, the overall explanatory power of the regression equation on the pricing bias is not good enough (R-squared = 0.688). To better understand pricing bias, much research remains to be done in the future.

6. Conclusions

We modeled an artificial European option market which has liquidity costs with an agent-based modeling and simulation approach inspired by the uncertain volatility approach in Wilmott et al. (1998). We observe that the option prices increase slightly when the parameter of liquidity costs increases. We also proved that there is significant pricing bias in the presence of unknown volatility. Moreover, the unknown drift has a significant nonlinear effect in the pricing bias. Finally, pricing bias tends to decrease as the drift increasing in the case of low volatility.

Despite its simplicity, we show that agent based simulation of the financial market provides a mechanism for conducting experiments that shed light on fundamental properties of the market price. Our approach may serve as a first step towards the goal of providing a methodology for option pricing with unknown volatility.

Acknowledgements

This research is funded by Vietnam National University HoChiMinh City (VNU-HCM) under grant number B2015-42-01.

References


